

Now It's Getting Complex . . . But It's Really Not Difficult!

2.6

Complex Number Operations

LEARNING GOALS

In this lesson, you will:

- Calculate powers of i .
- Interpret the real numbers as part of the complex number system.
- Add, subtract, and multiply complex numbers.
- Add, subtract, and multiply complex polynomial expressions.
- Understand that the product of complex conjugates is a real number.
- Rewrite quotients of complex numbers.

KEY TERMS

- the imaginary number i
- principal square root of a negative number
- set of imaginary numbers
- pure imaginary number
- set of complex numbers
- real part of a complex number
- imaginary part of a complex number
- complex conjugates
- monomial
- binomial
- trinomial

“Let me hear the downbeat!” might be something you hear the lead singer tell the band to start a song during a performance. In fact for centuries, bands, ensembles, barber shop quartets, and orchestras relied on tempo and beats to sync up with other band members. Well, this is true for band members today, but there is also music that doesn't have any band members—but a single musician mixing it up on turntables or on a laptop! Of course, these solo musicians are called DJs who have been mixing it since the late 1960s.

The cornerstone of almost any DJ's music is the art of sampling. Sampling is taking a portion or a “sample” of one sound recording and repurposing it into another song. One of the most common samples is taking the drum beats. Many DJs will take four measures of drum beats (with each measure having 4 beats per measure), and reuse it to become the spine of their new piece. Sometimes those four drum-beat measures are repeated for an entire piece—and sometimes these pieces can last 20 to 30 minutes in duration with the DJ infusing other samples of vinyl noise, ambient sound effects, record scratches, and lyrics.

Even more recently, artists have been using technology to create mashups. Mashups generally use two or more pre-recorded songs (not just samples, but entirely mixed songs) and arranging them together to create a new musical piece. Do you think that mashups use this same concept of 4 measures of music to create new musical pieces? Why do you think “4” is so special in creating music?

PROBLEM 1 The Powers of i

So far within this course, you have worked within the set of real numbers and determined real number solutions. Remember, the set of real numbers includes the sets of rational and irrational numbers.

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1. Consider the equation $x^2 = -1$.
Is there a real number solution to this equation?
Explain why or why not.

So if it's not a real number, does that mean it's a fake number?



The **imaginary number i** is a number such that $i^2 = -1$. Because no real number exists such that its square is equal to a negative number, the number i is not a part of the real number system.

2. If $i^2 = -1$, what is the value of i ?



3. Use the values of i and i^2 and the properties of exponents to calculate each power of i . Enter your results in the table and show your work.

Powers of i			
$i =$	$i^2 =$	$i^3 =$	$i^4 =$
$i^5 =$	$i^6 =$	$i^7 =$	$i^8 =$
$i^9 =$	$i^{10} =$	$i^{11} =$	$i^{12} =$

Use previously calculated powers of i to calculate the next power of i .



4. Describe any patterns you see in the table.

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5. Tristan, Kira, and Libby calculated the power i^{15} using different methods as shown.
 a. Explain why each student's method is correct.

Tristan

$$\begin{aligned}
 i^{15} &= (i^4)^3 \cdot i^3 \\
 &= (1)^3(-\sqrt{-1}) \\
 &= 1(-\sqrt{-1}) \\
 &= -\sqrt{-1}
 \end{aligned}$$

What's so special about multiplying by i^4 ?



Kira


The exponent of i^{15} is 15. When I divide 15 by 4, I have a remainder of 3. I know $i^3 = -\sqrt{-1}$. So, $i^{15} = i^3 = -\sqrt{-1}$.

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Libby

$$\begin{aligned}
 i^{15} &= i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \\
 &= \underbrace{\sqrt{-1} \cdot \sqrt{-1}}_{-1} \cdot \underbrace{\sqrt{-1} \cdot \sqrt{-1}}_{-1} \cdot \underbrace{\sqrt{-1} \cdot \sqrt{-1}}_{-1} \cdot \underbrace{\sqrt{-1} \cdot \sqrt{-1}}_{-1} \cdot \underbrace{\sqrt{-1} \cdot \sqrt{-1}}_{-1} \cdot \sqrt{-1} \\
 &= -1 \cdot \sqrt{-1} \\
 &= -i\sqrt{-1}
 \end{aligned}$$

- b. If you had to calculate i^{99} , whose method would you use and why?

-  6. Explain how to calculate any integer power of i .



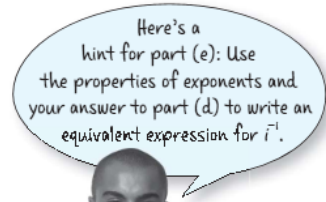
7. Calculate each power of i .

a. $i^{93} =$

b. $i^{206} =$

c. $i^{400} =$

d. $i^{-2} =$



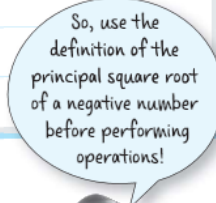
Here's a hint for part (e): Use the properties of exponents and your answer to part (d) to write an equivalent expression for i^{-1} .

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Now that you know about i , you can rewrite expressions involving negative roots. For any positive real number n , the **principal square root of a negative number**, $-n$, is defined by $\sqrt{-n} = i\sqrt{n}$.

Determine the value of $\sqrt{-75}$.

$$\begin{aligned} \sqrt{-75} &= \sqrt{(-1)(75)} \\ &= \sqrt{-1} \cdot \sqrt{75} \\ &= i\sqrt{25 \cdot 3} \\ &= 5\sqrt{3}i \end{aligned}$$

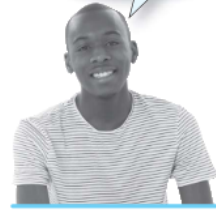


So, use the definition of the principal square root of a negative number before performing operations!

8. Analyze Georgette's work.

Georgette

$$\begin{aligned} \sqrt{-32} &= -\sqrt{32} \\ &= -\sqrt{16(2)} \\ &= -4\sqrt{2} \end{aligned}$$



Explain why she is incorrect.

9. Jen and Tami each rewrote the expression $\sqrt{-4} \cdot \sqrt{-4}$.



Jen

$$\begin{aligned} &\sqrt{-4} \cdot \sqrt{-4} \\ &= \sqrt{(-4)(-4)} \\ &= \sqrt{16} \\ &= 4 \end{aligned}$$

Tami

$$\begin{aligned} &\sqrt{-4} \cdot \sqrt{-4} \\ &= 2i \cdot 2i \\ &= 4i^2 \\ &= -4 \end{aligned}$$

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Who's correct? Explain the error in the other student's reasoning.



10. Rewrite each expression using i .

a. $\sqrt{64} - \sqrt{-63} =$

b. $\sqrt{-13} + 10 =$

c. $\frac{1 - \sqrt{-44}}{2} =$

PROBLEM 2 I Must Have Only Imagined this Was Complex

The **set of imaginary numbers** is the set of all numbers written in the form $a + bi$, where a and b are real numbers and b is not equal to 0. A **pure imaginary number** is a number of the form bi , where b is not equal to 0.

Why should I care about numbers that are imaginary?



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- Write the imaginary number i in the form $a + bi$. What are the values of a and b ?

- Give an example of a pure imaginary number.

Imaginary numbers actually have applications in real life. They are used in the scientific fields of electromagnetism, fluid dynamics, and quantum mechanics, just to name a few!

- Can a number be both real and imaginary? Explain why or why not.



The **set of complex numbers** is the set of all numbers written in the form $a + bi$, where a and b are real numbers. The term a is called the **real part of a complex number**, and the term bi is called the **imaginary part of a complex number**.



- Identify whether each number is a complex number. Explain your reasoning.

a. i

b. 3

c. -5.5216

d. $\pi + 3.2i$

5. What is the difference between a complex number and an imaginary number?

6. Create a diagram to show the relationship between each set of numbers shown.

- complex numbers
- imaginary numbers
- integers
- irrational numbers
- natural numbers
- rational numbers
- real numbers
- whole numbers



7. Use the word box to complete each statement. Explain your reasoning.

always	sometimes	never
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- a. If a number is an imaginary number, then it is _____ a complex number.
- b. If a number is a complex number, then it is _____ an imaginary number.
- c. If a number is a real number, then it is _____ a complex number.
- d. If a number is a real number, then it is _____ an imaginary number.
- e. If a number is a complex number, then it is _____ a real number.



PROBLEM 3 Call the Doctor, Stat! It's Time to Operate!

You know how to perform the basic operations of addition, subtraction, multiplication, and division on the set of real numbers. You can also perform these operations on the set of complex numbers.

When operating with complex numbers involving i , combine like terms by treating i as a variable (even though it is a constant).

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1. Simplify each expression. Show your work.

a. $(3 + 2i) - (1 - 6i) =$

b. $4i + 3 - 6 + i - 1 =$

c. $5i(3 - 2i) =$

d. $(5 + 3i)(2 - 3i) =$

2. Determine each product.

a. $(2 + i)(2 - i) =$

b. $\left(\frac{1}{2} + i\right)\left(\frac{1}{2} - i\right) =$

c. $(3 + 2i)(3 - 2i) =$

d. $(1 - 3i)(1 + 3i) =$



e. What do you notice about each product?



You may have noticed that the products in Question 2 did not contain an imaginary number, even though the original expression contained imaginary numbers. Each pair of expressions in Question 2 are called *complex conjugates*.

Complex conjugates are pairs of numbers of the form $a + bi$ and $a - bi$. The product of a pair of complex conjugates is always a real number and equal to $a^2 + b^2$.

Remember that a polynomial is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients. The definition of a polynomial can now be extended to include imaginary numbers.

A polynomial in one variable is an expression of the form $a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, where the coefficients (a_0, a_1, a_2, \dots) are complex numbers (real or imaginary) and the exponents are nonnegative integers.

A polynomial can have a special name, according to the number of terms it contains. A polynomial with one term is called a **monomial**. A polynomial with two terms is called a **binomial**. A polynomial with three terms is called a **trinomial**.

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3. Maria says that the expression $3x + xi - 5$ is a trinomial because it has three terms. Dante says that the expression is a binomial because it can be rewritten as the equivalent expression $(3 + i)x - 5$, which has two terms. Jermaine says that it is not a polynomial. Who is correct? Explain your reasoning.



4. Identify each expression as a monomial, binomial, trinomial, or other. Explain your reasoning.

a. $3 + 5i$

b. $-4xi + 2x - 5i + 1$

c. $\frac{3}{2}x^2 - \frac{1}{4}x^2i$

d. $1.5x + 3i + 0.5x^3i$

You can simplify some polynomial expressions involving i using methods similar to those you used to operate with numerical expressions involving i .

5. Simplify each polynomial expression, if possible. Show your work.

a. $xi + xi =$

b. $xi + xy =$

c. $-2.5x + 3i - xi + 1.8i + 4x + 9 =$

d. $(x + 3i)^2 =$

e. $(2i - 4x)(i + x) =$

You just need to remember the rules for multiplying two binomials.



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6. Analyze each method. Explain each student's reasoning. Then, identify which of the two methods seems more efficient and explain why.

a.

 **Shania**

$$\begin{aligned}(2 - i)(1 + 2i)(2 + i) &= (2 - i)(2 + i)(1 + 2i) \\ &= (4 - i^2)(1 + 2i) \\ &= (4 - (-1))(1 + 2i) \\ &= 5(1 + 2i) \\ &= 5 + 10i\end{aligned}$$

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 **Lindsay**

$$\begin{aligned}(2 - i)(1 + 2i)(2 + i) &= (2 + 3i - 2i^2)(2 + i) \\ &= (2 + 3i - 2(-1))(2 + i) \\ &= (4 + 3i)(2 + i) \\ &= 8 + 10i + 3i^2 \\ &= 8 + 10i + 3(-1) \\ &= 5 + 10i\end{aligned}$$

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b.

 **Elijah**

$$\begin{aligned}
 &(x + i) + (x + 3) + (x + 3i)(x + 3) \\
 &= (x^2 + 3x + xi + 3i) + (x^2 + 3x + 3xi + 9i) \\
 &= (x^2 + x^2) + (3x + 3x) + (xi + 3xi) + (3i + 9i) \\
 &= 2x^2 + 6x + 4xi + 12i
 \end{aligned}$$

 **Aiden**

$$\begin{aligned}
 (x + i) + (x + 3) + (x + 3i)(x + 3) &= (x + 3)(x + i + x + 3i) \\
 &= (x + 3)(2x + 4i) \\
 &= 2x^2 + 4xi + 6x + 12i
 \end{aligned}$$



7. Simplify each expression.

a. $(2 - i)(1 + 2i)(2 + i)$

b. $(x + i)(x + 3) + (x + 3i)(x + 3)$

PROBLEM 4 Rewriting Quotients of Complex Numbers

Division of complex numbers requires the use of complex conjugates, thus changing the divisor into a real number. Recall, the complex conjugate of $a + bi$ is $a - bi$.



1. For each complex number, write its conjugate. Then calculate each product.

a. $7 + i$

b. $-5 - 3i$

Remember that
 $(a + bi)(a - bi) = a^2 + b^2$.

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c. $12 + 11i$

d. $-4i$



You can rewrite the division of a complex number by multiplying both the divisor and the dividend by the conjugate of the divisor, thus changing the divisor into a real number.



You can rewrite $\frac{3 - 2i}{4 + 3i}$ without a complex number in the denominator.



$$\frac{3 - 2i}{4 + 3i} = \frac{3 - 2i}{4 + 3i} \cdot \frac{4 - 3i}{4 - 3i}$$



$$= \frac{12 - 9i - 8i + 6i^2}{4^2 + 3^2}$$



$$= \frac{12 - 17i - 6}{16 + 9}$$



$$= \frac{6 - 17i}{25}$$



$$\frac{3 - 2i}{4 + 3i} = \frac{6}{25} - \frac{17i}{25}$$



You are just
multiplying by a
form of 1.





2. Rewrite each quotient without a complex number in the denominator.

a. $\frac{2 - i}{3 + 2i}$

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b. $\frac{3 - 4i}{2 - 3i}$

c. $\frac{5 + 2i}{1 + i}$

d. $\frac{20 - 5i}{2 - 4i}$



Be prepared to share your solutions and methods.